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MULTICOMPONENT STEADY FLOWS IN POROUS MEDIA WITH PHASE TRANSITIONS†

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Steady multiphase flow of a multicomponent mixture in a porous medium with phase transitions is considered. It is shown that, in a wide class of cases, the thermodynamic problem separates from the filtration problem and the latter is integrated in quadratures. The class of exact solutions which has been found is used to interpret indicator curves. Solutions are presented in an analytic form for systems of the "gas–condensate" and "oil–gas" type.

1. Suppose there is a steady flow of a multicomponent mixture in a homogeneous isotropic porous medium with a permeability k. We shall assume that the process is isothermal. Let T be the corresponding temperature, let N be the number of components in the reservoir mixture and let K be the number of phases. We shall consider a class of flows which are characterized by a single spatial coordinate r which includes one-dimensional ($\nu = 0$), cylindrically symmetric ($\nu = 1$) and spherically symmetric ($\nu = 2$) flows. In the latter two cases, r is a radial coordinate.

It is well known from the chemical thermodynamics [1, 2] that all the thermodynamic properties of a mixture can be found from the free energy per unit volume of the mixture $f = f(T, n_i)$, where n_i is the density of molecules of the ith component in the mixture. The entropy per unit volume s, the pressure p and the chemical potentials κ_i are connected by the relation

$$df = -sdT + \sum_{i=1}^{N} \kappa_i dn_i, \quad f = -p + \sum_{i=1}^{N} \kappa_i n_i \tag{1.1}$$

The Duhem equality

$$0 = sdT - dp + \sum_{i=1}^{N} n_i d\kappa_i \tag{1.2}$$

follows from (1.1)

The state of a mixture in an element of the porous medium is characterized by a set of quantities n_{ij}, s_j, where n_{ij} is the density of molecules of the ith component in the jth phase and s_j is the saturation of the pore space by the jth phase. Henceforth $i, k = 1, \ldots, N; j, l = 1, \ldots, K$.

In the formulation being considered the filtration equations [3] reduce to equations for the conservation of the fluxes of the components

$$\frac{d}{dr}\left(r^{\nu} k \sum_{j=1}^{N} \frac{n_{ij} f_j}{\mu_j} \frac{dp_j}{dr} \right) = 0 \tag{1.3}$$

where μ_j is the viscosity of the jth phase and f_j is the phase permeability of the jth phase which is a function of the saturations.

It is convenient to introduce the new coordinate

$$\eta = \int_a^r x^{-\nu} dx \tag{1.4}$$

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where a is a certain fixed constant with the dimensions of length.

Equations (1.3) have the first integrals

$$\sum_{j=1}^{N} A_j n_{ij} = \frac{Q_i}{k\alpha_\nu}, \quad A_j = f_j \mu_j^{-1} \frac{dp_j}{d\eta} \tag{1.5}$$

$$\alpha_0 = 1, \quad \alpha_1 = 2\pi, \quad \alpha_2 = 4\pi$$

where Q_i is the flow of molecules of the ith component (this is taken per unit cross-sectional area when $\nu = 0$ and per unit length when $\nu = 1$).

We shall assume the well-known expressions

$$\mu_j = \mu_j(T, n_{il}), \quad f_j = f_j(s_l), \quad p_j = p(T, n_{il}) \tag{1.6}$$

In accordance with Gibb's phase rule, we shall assume that $K \leqslant (N + 1)$.

The problem consists of determining $(KN + K - 1)$ unknown functions n_{ij}, s_j.

The problem becomes closed if $N(K - 1)$ phase equilibrium equations

$$\kappa_i(T, n_{k1}) = \ldots = \kappa_i(T, n_{kK}) \tag{1.7}$$

and, also, $(K - 1)$ equations for the capillary pressure jumps

$$p_{m+1} - p_m = p_{cm}, \quad m = 1, \ldots, (K - 1) \tag{1.8}$$

where $p_{cm} = p_{cm}(s_l)$ are specified functions, are added to the N equations (1.5).

Let us investigate the system of equations (1.5) in greater detail by considering this system as the problem of determining A_j, n_{ij} in the case of a specified right-hand side with the supplementary conditions (1.7) and (1.8).

When a solution exists, the following alternatives may be formulated: (A) the N-dimensional vectors $\mathbf{n}_j = (n_{ij})$ are linearly independent (and, consequently, $K \leqslant N$) or (B) the N-dimensional vectors $\mathbf{n}_j = (n_{ij})$ are linearly dependent.

In case A, the quantities A_j are uniquely defined. Let a certain A_{j0} be equal to zero and the j_0th phase have a non-trivial mobility threshold. The filtration problem is then non-uniquely solvable since it is possible to find a solution for an arbitrarily specified saturation distribution $s_{j0}(\eta)$ below the mobility threshold. If the mobility threshold is trivial, then $s_{j0} = 0$.

For the pressures, we obtain the system of equations

$$f_j dp_j / d\eta = A_j \mu_j \tag{1.9}$$

We shall show that this system does not generate second-order relations from the first-order relations (1.6) and (1.7). Note that, according to (1.2)

$$(\partial p / \partial \kappa_i)_T (T, n_{ij}) = n_{ij}$$

It therefore follows from (1.7) and (1.9) that

$$f_j \sum_{i=1}^{N} n_{ij} \frac{d\kappa_i}{d\eta} = A_j \mu_j \tag{1.10}$$

By virtue of the linear independence of the N-dimensional vectors n_{ij}, system (1.10) always admits of a certain solution for the quantities $d\kappa_i/d\eta$ and, consequently, also for the quantities $dn_{ij}/d\eta$. No new functional constraints on the unknown functions arise.

In case B, the quantities A_j are non-uniquely defined. This non-uniqueness is eliminated if the A_j are chosen in such a manner that system (1.10) admits of a certain solution for the quantities $d\kappa_i/d\eta$, that is, in such a manner that second-order relations do not arise.

Finally, if just a single solution exists in the case of system (1.5), then a solution of the filtration problem exists (at least, locally with respect to η) which satisfies the system of equations (1.9) and (1.8). Equations (1.9) are non-trivial only in the case of non-zero coefficients A_j.

It is natural to raise the question: what right-hand sides in Eqs (1.5) ensure that a solution will exist? That is, what component fluxes can be specified for the corresponding flow in a porous medium to exist?

Generally speaking, as Q_i, one may take any N-dimensional vector from the linear envelope of an arbitrary set of N-dimensional vectors $\mathbf{n}_j = (n_{ij})$ related by the phase equilibrium conditions (1.7). It is, however, difficult to construct this linear envelope in explicit form.

An important class of cases exists when the thermodynamic problem of calculating the phase equilibrium separates out from the filtration problem and when it is possible to assign a clear physical meaning to the permitted right-hand sides in Eqs (1.5).

Let us assume that the component fluxes are positive: $Q_i > 0$ and introduce the following notation: $Q = \Sigma_{i=1}^{N} Q_i > 0$ is the total flow, $n_j = \Sigma_{i=1}^{N} n_{ij}$ is the total density for the phase with number j, and $c_{ij} = n_{ij}/n_j$ is the concentration of the ith component in a phase with number j, $q = Q/(k\alpha_v)$, $B_j = n_jA_j/q$, $c_{i0} = Q_i/Q$.

Equations (1.5) reduce to the form

$$\sum_{j=1}^{K} B_j c_{ij} = c_{i0} \tag{1.11}$$

The form of system (1.11) enables us to interpret the coefficients B_j as the molar fractions of the phases in the case of the decomposition of a certain hypothetical mixture of composition c_{i0} into K phases with compositions c_{ij} and pressures in the phases p_j. The problem of decomposition of this mixture can be solved, in principle, independently of the filtration problem: either by carrying out a calculation using semi-empirical equations of state [4–6] or by creating the mixture experimentally and investigating its decomposition in capillaries with different radii. A family of relations $B_j = B_j(T, p_i)$ is the result of the solution of the thermodynamic problem.

It is convenient to introduce the coefficients $C_j = B_j n^{-1}\mu_j$. As previously, if, for a certain j_0, the corresponding quantity C_{j0} is equal to zero and, here, the j_0th phase has a non-trivial mobility threshold, the filtration problem is non-uniquely solvable, since it is possible to find a solution in the case of an arbitrarily specified saturation distribution $s_{j0}(\eta)$ below the mobility threshold. If the mobility threshold is trivial than $s_{j0} = 0$. If one introduces the new independent variable $\xi = Q\eta/k$, then system (1.9) is rewritten in a form which does not contain an explicit dependence on the total flux and permeability

$$f_j dp_j / d\xi = \alpha_v^{-1} C_j \tag{1.12}$$

Hence, the problem reduces to a system of ordinary differential equations in p_j with relations (1.8) for finding the saturations s_j. This system is autonomous since, if just a single solution $p_{j0}(\xi)$, $s_{j0}(\xi)$ is known, then all the remaining solutions can be obtained by a shift transformation of the argument: $p_j = p_{j0}(\xi + \xi_0)\, s_j = s_{j0}(\xi + \xi_0)$. The latter is equivalent to a change in the parameter a in formula (1.4).

We note that, if we certainly know that a mixture is in a single phase state at a certain point, then the composition c_{i0} is to be identified with the composition of the mixture at this point.

We will now consider the case when capillary forces can be neglected. The pressure $p = p(\eta)$ is then defined as being the same for all phases.

It is seen from (1.5) that the formula

$$c_{i0} = \sum_{j=1}^{K} \frac{n_{ij} f_j}{\mu_j} \left(\sum_{j=1}^{K} \frac{n_j f_j}{\mu_j} \right)^{-1} \tag{1.13}$$

holds.

Hence, if the state of a reservoir mixture is known at just a single point $r = r_d$ then the quantities c_{i0}

are determined using formula (1.13).

System (1.12) reduces to the equation

$$\sum_{j=1}^{K} f_j \frac{dp}{d\xi} = \alpha_\nu^{-1} \sum_{j=1}^{K} C_j \tag{1.14}$$

The saturations are determined from the proportion

$$C_1:\ldots:C_K = f_1:\ldots:f_K. \tag{1.15}$$

Hence, when there are no capillary forces, the problem reduces to the single ordinary differential equation (1.14) and the family of relations (1.15).

2. We will now consider the question of how the above theory can be applied to steady-state well tests. We shall assume that the flow close to a well is cylindrically symmetric ($\nu = 1$).

Now, suppose we know how the driving pressure p_w depends on the production rate D of the well, expressed in moles: $p_w = F_0(D)$ for a productive layer of thickness h_0.

Steady-state well tests (indicator curves) are customarily used to find the permeability. If the phase permeabilities and all the properties of the reservoir mixture are known a priori, then system (1.8), (1.12) can be solved. Let us assume that a solution $p_{j0}(\xi)$, $s_{j0}(\xi)$ can be chosen which corresponds to flows that are realized when plotting the indicator curve. The pressure on the face p_w is to be identified with the pressure in the phase which constitutes the main production of the well (with p_1, for example) at a value of the parameter $\eta = \eta_{w0}$ which corresponds to the well radius r_{w0}. So

$$F_0(D) = p_{10}(D(k_0h_0)^{-1}\eta_{w0}) \tag{2.1}$$

Relations (2.1) must be satisfied for a certain value k_0 in the specified range of production rates D. Hence, (2.1) is an overdetermined system of equations for finding k_0.

In practice, it can hardly be expected that the experimental values $F_0(D)$ and the calculated values on the right-hand side of (2.1) will be identical, due to the incompleteness of the information regarding the phase permeabilities and properties of the reservoir mixture. One can only hope to achieve closeness of the left- and right-hand sides of (2.1) by selecting k_0 (in the sense of the mean square deviation, for example).

The theory described in Section 1 enables us to draw a number of conclusions without solving system (1.8), (1.12).

Let the indicator curve $p_w = F_0(D)$ be known for a reservoir with a known thickness parameter h_0 and permeability k_0. We will get show how, knowing this curve, it is possible, by a simple scaling of the argument, to obtain the indicator and depression curves for all wells and reservoirs with the same reservoir parameters c_{i0}.

First, we find the depression curve from formula (2.1)

$$p_{10}(\xi) = F_0((k_0h_0)\xi\eta_{w0}^{-1}) \tag{2.2}$$

Next, for a certain well, let r_{d1} be the drainage radius, r_{w1} be the well radius, p_{d1} be the reservoir pressure (that is, the pressure when $r = r_{d1}$), k_1 be the permeability, and h_1 be the depth of the layer. Let us define the quantities ξ_0, ξ_1, ξ_2 by the formulae

$$p_{i0}(\xi_0) = p_{d1}, \quad \xi_1 = \frac{D}{k_1h_1}\ln\frac{r}{r_{d1}}, \quad \xi_2 = \frac{D}{k_1h_1}\ln\frac{r_{w1}}{r_{d1}}$$

Then, $p_{11} = p_{10}(\xi_1 + \xi_0)$ is the new depression curve and $p_w = p_{10}(\xi_2 + \xi_0)$ is the new indicator curve. Comparison of the calculated experimental curves (2.2) can serve to refine the parameters of the reservoir mixture and the phase permeabilities.

3. Generally speaking, when the expressions specifying the phase permeabilities and properties of a reservoir mixture have a sufficiently general form, system (1.8), (1.12) can only be solved by numerical methods. However, in a number of cases, simplifying assumptions enable one to obtain approximate solutions in analytic form.

Let us consider the flow of a two-phase mixture of a "gas–hydrocarbon liquid" kind. We will assume that the mixture is in a single phase state at sufficiently high pressures beyond the well.

Far from the well at pressures which are close to the initial reservoir pressure p_0, the mixture is in a single phase state. Then, according to formula (1.13), c_{i0} are the concentrations of the components of the reservoir mixture.

Hence, Eqs (1.1) can be interpreted as the conditions which relate the composition c_{ig} of the gas phase and the composition c_{il} of the liquid phase to the mean composition c_{i0} in the system in which the gas and liquid phases coexist in thermodynamic equilibrium at the corresponding pressures p_g and p_l. Let W be the molar fraction of the liquid phase in this system, s_g be the saturation of the gas phase, and s_l be the saturation of the liquid phase. Equations (1.11) reduce to the form

$$(1-W)c_{ig} + Wc_{il} = c_{i0} \tag{3.1}$$

Since the right-hand sides of Eqs (3.1) are constant quantities, W is a certain function of the pressures p_g and p_l: $W = W(p_g, p_l)$. In principle, this function can be determined if the equilibrium equations are solved or if laboratory investigations are carried out on a sample of the reservoir mixture. In this case, system (1.8), (1.12) has the form

$$\begin{gathered} f_g dp_g / d\xi = (1-W)\alpha_\nu^{-1}\mu_g n_g, \quad f_l dp_l / d\xi = W\alpha_\nu^{-1}\mu_l n_l \\ p_q - p_l = p_c \end{gathered} \tag{3.2}$$

and serves to determine the unknowns s_q, p_q, p_l.

If the gas (liquid) phase has a non-trivial mobility threshold $s_{g*} > 0$ ($s_{l*} > 0$), two situations are possible when $W = 1$ ($W = 0$): either the given phase is not present $s_q = 0$ ($s_l = 0$) or it is present and $s_g < s_{g*}$ ($s_l < {}_l{}^*$). Regions of two-phase filtration are therefore possible when one phase is mobile and has the composition c_{i0} and the other phase is immobile. As was shown in Section 1, this leads to the non-uniqueness of the solution. Such solutions with immobile phases are unfavourable from the point of view of the exploitation of liquid hydrocarbon deposits since they yield a greater depression at the same production rate. The conditions under which such solutions arise require further analysis.

Let us now focus on solutions in which there are no immobile phases. This means that, in the case of a non-trivial mobility threshold, the appearance of a new phase is accompanied by a discontinuity in the saturation.

If the pressure in the gas phase p_g is fixed and the behaviour of the solutions in the two-phase region with a bounded ratio (dp_g/dp_l) when $\varepsilon = \mu_g/\mu_l \to 0$ is considered, then, as follows from (3.2), the saturation s_g tends to a value s_{g*} which corresponds to the mobility threshold for the gas phase.

Let us solve system (3.2) for the case when $p_c = 0$. Here, $p_g = p_l = p$ and system (3.2) reduces to a single ordinary differential equation

$$\alpha_\nu(\mu_g^{-1}(n_g f_g) + \mu_l^{-1}(n_l f_l))dp / d\xi = 1 \tag{3.3}$$

and to the equation for determining the saturation s_g

$$W:(1-W) = (f_l n_l \mu_g):(f_g n_g \mu_l) \tag{3.4}$$

Let us now consider some actual situations.

An oil deposit with dissolved gas. In this case c_{i0} is the composition of the oil in the reservoir. Let us assume that the density of the liquid phase n_l is a constant quantity

$$n_l = n_{l0} = \text{const} \tag{3.5}$$

that is, degassing and reducing the pressure have a negligible effect on the density of the liquid phase. Let us also assume that

$$\mu_q = \text{const}, \ \ \mu_l = \text{const} \tag{3.6}$$

Next, let p_t be the saturation pressure. For the molar fraction of the liquid, we take the piecewise-linear dependence

$$W = 1, \ \ p > p_t; \ \ W = 1 - \beta(p_t - p), \ \ p \leqslant p_t, \ \ \beta > 0 \tag{3.7}$$

We shall seek a solution of problem (3.3), (3.4) in the lowest approximation with respect to the small parameter $\varepsilon = \mu_g/\mu_l$ with the boundary condition

$$p(\xi_d) = p_d > p_t \tag{3.8}$$

As was indicated above, here, in the two-phase region

$$s_g = s_{g*} + \Delta s, \quad \Delta s \ll 1 \tag{3.9}$$

$$f_l = f_0 + O(\Delta s), \quad f_0 = f_l(s_{g*})$$

The equation

$$dp / d\xi = \mu_l(\alpha_v n_{l0})^{-1}$$

with the obvious solution

$$p = p_t + \mu_l(\alpha_v n_{l0})^{-1}(\xi - \xi_t), \ \ \xi_t = p_d - (p_d - p_t)\mu_l^{-1}(\alpha_v n_{l0})$$

is obtained from (3.3)–(3.9) in the single-phase region $p > p_l$.

In the two-phase region $p \leqslant p_t$, the equation

$$(1 - \beta(p_t - p))^{-1} dp / d\xi = \mu_l(\alpha_v n_{l0} f_0)^{-1}$$

is obtained which has the solution

$$p = p_t + \beta^{-1}(\exp(\beta\mu_l(\alpha_v n_{l0} f_0)^{-1}(\xi - \xi_t)) - 1)$$

A gas condensate deposit. In this case c_{i0} is the composition of the reservoir gas.

In order to solve Eqs (3.3) and (3.4), it is necessary to specify the values of W, n_g, n_l, μ_g, μ_l. Experiments, as well as calculations using semi-empirical equations of state [4–6] show that the molar fraction of the liquid phase at a pressure below the pressure at the onset of condensation p_l and a temperature T above the critical temperature is approximated quite well by the expression

$$W = a_1 p + a_2/(p - a_3)$$

where a_1, a_2, a_3 are positive quantities which depend on T and c_{i0}. These quantities can be found if the pressure P_*, at which W reaches its maximum value W_*, is known. The three equations for finding the quantities a_1, a_2, a_3

$$W|_{p=p_t} = 0, \ \ dW / dp|_{p=p_*} = 0, \ \ W|_{p=p_*} = W_*$$

have two sets of solutions when $p_*/p_t > 0.75$ (this condition is well satisfied in the case of natural mixtures). The best approximation is achieved in the case of the solution with the lower value of the parameter a_3.

Following the tradition adopted in the calculation of processes in gas condensate deposit, for the gas phase we take the equation of state

$$p = Zk_BTn_g$$

where Z is a constant which is called the coefficient of overcompressibility and k_B is Boltzmann's constant. We shall also adopt assumptions (3.5), (3.6) and (3.8)

In the region of two-phase flow where, as was shown earlier, $s_l > s_{l*}$ and the quantity s_g is close to s_{g*}, we shall take the linear dependence for the phase permeabilities

$$f_g = f_{g1}s_g + f_{g0}, \quad f_l = f_{l1}s_l + f_{l0}$$

The density n_{l0} can be associated with the quantity with the dimensions of pressure $p_{l0} = k_BZTn_{l0}$. When $p > p_l$, there is no liquid phase ($W = 0$), and the problem has the well-known solution

$$p^2 - p_d^2 = 2\mu_l k_B TZ(\xi - \xi_d)\alpha_v^{-1} \tag{3.10}$$

When $p \leqslant p_1$, the ordinary differential equation

$$dp / d\xi = \mu_l p_{l0}(n_{l0}p\alpha_v)^{-1}\Gamma F$$

$$\Gamma = (f_{l1}f_{g1} + f_{l1}f_{g0} + f_{l0}f_{g1})^{-1}, \quad F = f_{l1}\varepsilon(1 - W) + f_{g1}pp_{l0}^{-1}W$$

is obtained for the pressure.

Analysis shows that the roots p_1, p_2, p_3 of the equation $F = 0$ can be: (1) different with p_1 and p_t, and (2) $p_1 > p_t, p_2 = p_3 < 0$. In the first case, the identity

$$\frac{p(p - a_3)}{(p - p_1)(p - p_2)(p - p_3)} = \sum_{i=1}^{3} \frac{A_i}{p - p_i}$$

holds, where A_i are certain coefficients. An implicit equation for p is derived from this identity

$$\sum_{i=1}^{3} A_i \ln\frac{p - p_i}{p_t - p_i} = \chi, \quad \chi = \mu_l a_1 n_{l0}^{-1}\Gamma f_{g1}\alpha_v^{-1}(\xi - \xi_t)$$

where ξ_t, which corresponds to p_d, is determined from (3.10).

In the second case, the identity

$$\frac{p(p - a_3)}{(p - p_1)(p - p_2)^2} = \frac{A_1}{p - p_1} + \frac{A_2}{p - p_2} + \frac{A_3}{(p - p_2)^2}$$

holds and, correspondingly, we have the solution

$$-\frac{A_3}{p - p_2} + \sum_{j=1}^{2} A_i \ln\frac{p - p_i}{p_t - p_i} = \chi - \frac{A_3}{p_t - p_2}$$

These solutions are matched at the point $p = p_l$ with the solution (3.10).

4. The problem of the steady multiphase flow of a multicomponent mixture in a porous medium with phase transitions is therefore integrable in quadratures for a wide class of situations. In this case, there is no need to solve the phase equilibrium equations at each point in order to find the pressure and saturation curves: it is sufficient to carry out laboratory experiments or thermodynamic calculations for a certain hypothetical mixture which can sometimes be identified with the initial reservoir mixture.

The solution of the filtration problem proposed in this paper enables one to make a direct forecast of the operation of a working well and to scale the characteristics of certain wells to other wells. Approximate solutions of the filtration problem can be given in analytic form in the case of two-phase systems of the "gas–oil" and "gas–condensate" type.

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